Acyclic Edge Colourings of Graphs with Large Girth

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ABSTRACT: An edge colouring of a graph G is called acyclic if it is proper and every cycle contains at least three colours. We show that for every $\varepsilon > 0$, there exists a $g = g(\varepsilon)$ such that if G has maximum degree Δ and girth at least g then G admits an acyclic edge colouring with $(1+\varepsilon)\Delta + O(1)$ colours.
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1. INTRODUCTION

An edge colouring of a graph $G = (V(G), E(G))$ is called *acyclic* if it is proper (two edges that share an end point have different colours) and every cycle contains at least three colours (otherwise stated, there are no bicoloured cycles). The acyclic chromatic index of G, denoted by $a'(G)$, is the minimum number of colours used in an acyclic edge colouring of G. Acyclic colourings were introduced by Grünbaum [10] in the context of vertex colouring of planar graphs (see [11] for more information on acyclic colourings).

Here we consider the problem of determining $a'(G)$ for a graph G with bounded maximum degree. It is well known, that a proper edge colouring of a graph G with maximum degree Δ may require $\Delta + 1$ colours. Fiamčik [7] and Alon, Sudakov and Zacks [2] independently conjectured that $a'(G) \leq \Delta + 2$. From an intuitive point of view, the conjecture states that at most one extra colour is needed to break all the bicoloured cycles of G. If true, the conjecture is best possible: there exists an infinite family of graphs G for which $a'(G) = \Delta + 2$ [2].

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Alon, McDiarmid and Reed [1] showed that $a'(G) \leq 64\Delta$. This result initiated a series of papers devoted to improve the constant. Molloy and Reed [14] showed that $a'(G) \leq 16\Delta$. This bound was improved by Ndreca, Procacci and Scoppola [19] to $a'(G) \leq \lceil 9.62(\Delta - 1) \rceil$ using a new version of the Lóvasz Local Lemma based on cluster expansion [5]. Esperet and Parreau [6] used the entropy compression method, based on the algorithmic version of the local lemma [17], to show that $a'(G) \leq 4\Delta - 4$.

The same problem has been studied in terms of the girth g of G. Alon, Sudakov and Zacks [2] showed that the acyclic edge colouring conjecture is true ($a'(G) \leq \Delta + 2$) if $g \geq c\Delta \log \Delta$, for some large constant c. They also noted that $a'(G) \leq 2\Delta + 2$ if $g \geq (1 + o(1)) \log \Delta$. Unfortunately, the previous conditions imply that these results are only valid if the girth goes to infinity when $\Delta \to +\infty$. Weaker results have been obtained in the case when the girth is not too large [6, 18, 19]. The best known result for large girth [8], states that if $g \geq 219$, then $a'(G) \leq \lceil 2.323(\Delta - 1) \rceil + 1$.

Recently, Bernshteyn [4] showed that for every $\varepsilon > 0$, G has acyclic chromatic index at most $(2 + \varepsilon)\Delta$ provided that the girth of G is large only with respect to ε. This result uses an Cut version of the Lovász Local Lemma that refines the entropy compression argument [3, 9].

Unfortunately, the standard entropy compression method seems to fail utterly if less than $2\Delta - 1$ colours are used. The heart of this method is a random recolouring procedure which recolours just one edge at a time. When applying such a procedure, we can deduce no pseudorandom properties of the colourings which we construct, so we have to allow for the possibility that the $2\Delta - 2$ edges adjacent to the edge that we are recolouring all have different colours.

In this paper we improve the previous result of Bernshteyn by providing an asymptotically tight upper bound using an iterative colouring procedure.

Theorem 1. *For every $\varepsilon > 0$, there exist constants $g = g(\varepsilon)$ and $\Delta_0 = \Delta_0(\varepsilon)$, such that if G has maximum degree $\Delta \geq \Delta_0$ and girth at least g, then*

$$a'(G) \leq (1 + \varepsilon)\Delta.$$

The proof of this theorem uses the iterative edge colouring procedure introduced by Kahn to provide a list edge colouring of a graph with $(1 + o(1))\Delta$ colours (see [12, 15] or Chapter 14 in [16]). Our main contribution is to track the partially coloured cycles in G that are still able to become bicoloured. In particular, we use the large girth condition to make sure that at each iteration there are many uncoloured edges in any such cycle. Since the number of iterations of our procedure depends only on ε, we need the girth to be large only with respect to ε.

We can restate the previous theorem in order that it holds for every Δ.

Corollary 2. *For every $\varepsilon > 0$, there exist constants $C = C(\varepsilon)$ and $g = g(\varepsilon)$ such that if G has maximum degree Δ and girth at least g,*

$$a'(G) \leq (1 + \varepsilon)\Delta + C.$$

From the proof of Theorem 1, one can easily derive an explicit lower bound on g in terms of ε; in particular, it suffices to set $g(\varepsilon) = \varepsilon^{-O(1)}$. We believe that this bound on g in terms of ε can be improved but probably a new approach is needed to get rid of the dependence on ε in the girth.

As far as we know, this is the first result on acyclic edge colourings of graphs with bounded girth that uses the asymptotically optimal number of colours. Finally, the proof of Theorem 1 can be easily adapted to deal with acyclic list edge colourings.

2. REDUCTION TO REGULAR GRAPHS

In this section we show that in order to prove Theorem 1 it suffices to do it for Δ-regular graphs. Although this is not a necessary step, it will simplify the proof of the main theorem.

Proposition 3. *If every Δ-regular graph with girth at least g admits an acyclic edge colouring with a colours, then every graph with maximum degree Δ and girth at least g also does.*

Proof. It is enough to show that we can embed G into a $\Delta(G)$-regular graph with the same girth. We do so now.

Suppose that $G_0 = G$ is not regular. We will construct a graph G_1 by considering many disjoint copies of G_0 and adding some edges between them. We will show that G_1 has a larger minimum degree while it has the same maximum degree and girth.

The r-th power a graph G is a graph which has the same vertex set of G, but in which vertices are adjacent when their distance in G is at most r. Let f_G be a colouring of the vertices of the g-th power of G_0 using the set of colours $\{1, \ldots, \Delta^g\}$. Since G_0 has maximum degree Δ, Δ^g colours suffice to obtain a proper colouring.

Let H be a Δ^g-regular bipartite graph with girth at least g. Since bipartite graphs are Class I (see Lemma 1.4.18 in [13]), we can obtain a Δ^g edge colouring f_H of H; that is each of the Δ^g colours is incident to every vertex in H.

Now consider G_1 composed of $|V(H)|$ many disjoint copies of G_0 where we add an edge between two copies of $v \in V(G_0)$ corresponding to vertices $u_1, u_2 \in V(H)$, if the degree of v in G_0 is smaller than Δ, $u_1u_2 \in E(H)$ and $f_H(u_1u_2) = f_G(v)$. The copy of G_0 in G_1 corresponding to $u \in V(H)$ will be denoted by G_0^u.

Observe that the graph G_1 has the same maximum degree as G_0 because we only add edges between vertices that have a degree smaller than Δ. Also the minimum degree has increased by 1 since for every vertex $u \in V(H)$ and every colour $c \in \{1, \ldots, \Delta^g\}$ there is a vertex $u' \in V(H)$ such that $uu' \in E(H)$ and $f_H(uu') = c$. Thus, every copy of a vertex that has degree smaller than Δ in G_0 is incident to a new edge in G_1.

Moreover, G_1 has girth at least g. Suppose that there is a cycle C of length ℓ which is strictly less than g. Since G_0 has no cycles of length ℓ, the cycle should contain vertices in different copies of G_0. In particular, the cycle induces a closed walk in H of length at least 2 and at most ℓ. If this walk contains a cycle, let C' be one of the minimal ones. Clearly the length of C' is at most ℓ, but H has girth at least g, a contradiction. Thus, we may assume that the walk is acyclic. Let $u \in V(H)$ be one of the leaves of the acyclic walk (there at least two of them) and let $u' \in V(H)$ be its unique neighbour in the closed walk. Otherwise stated, the cycle at some point enters in a vertex $v_1 \in V(G_0^u)$ from $G_0^{u'}$, stays in G_0^u for a while and exits from $v_2 \in V(G_0^u)$ towards $G_0^{u'}$. Since C has length ℓ, there exists a path in G_0^u from v_1 to v_2 of length at most ℓ. Moreover, since v_1 and v_2 have edges to $G_0^{u'}$, $f_G(v_1) = f_H(uu') = f_G(v_2)$. Hence, there are two vertices from G_0 at distance less than g with the same colour, obtaining a contradiction with the construction of f_G.

Repeating the same argument at most Δ times, we embed the graph G in a Δ-regular graph G_Δ with girth at least g. ∎

3. OUTLINE OF THE PROOF OF THEOREM 1

In this section we sketch the proof of Theorem 1.

Throughout the proof, we will assume that $\varepsilon \leq \varepsilon_0 = 10^{-3}$. Observe that if we show Theorem 1 for any $\varepsilon \leq \varepsilon_0$, then it also holds for all $\varepsilon > 0$ by setting $g(\varepsilon) = g(\varepsilon_0)$ and $\Delta_0(\varepsilon) = \Delta_0(\varepsilon_0)$ if $\varepsilon \geq \varepsilon_0$. For the sake of clarity of the presentation, we will omit floors and ceilings whenever they are not relevant for the argument. We use the notations $O(\cdot), o(\cdot)$, and $\Omega(\cdot)$ for classes of functions of Δ in the usual sense. We also use $O(1)$ to denote an implicit constant that does not depend on Δ but could depend on ε. Unless explicitly stated otherwise, all logarithms are natural. We use $V(G)$ and $E(G)$ to denote the vertex and the edge set of G. Finally, for every vertex v, we use $N(v)$ to denote the set of neighbours of v.

Our proof is an analogue of the one for list chromatic index in Chapter 14 of [16] where at the same time we control how many edges are coloured in partially bicoloured cycles. We will use the same notations as in [16] and mimic some of the arguments displayed there. Recall that by Proposition 3, we can assume that the graph G is Δ-regular. This allows us to simplify some technicalities in the proof of Theorem 1.

We will prove the following three lemmas from which the desired result follows. The first lemma says that at the beginning we can reserve a small number of colours for each vertex, satisfying some useful properties. We are going to use these colours at the end.

Lemma A. *There exists a collection $\{S_v : v \in V(G)\}$ of subsets of $[(1+\varepsilon)\Delta]$ such that*

(A.1) for every vertex v, $|S_v| \leq \frac{4\varepsilon\Delta}{9}$,
(A.2) for every edge $e = uv$, $|S_u \cap S_v| \geq \frac{\varepsilon^2\Delta}{18}$, and
(A.3) for every vertex v and every colour c, $|\{u \in N(v) : c \in S_u\}| \leq \frac{\varepsilon\Delta}{2}$.

The second lemma is the core of the proof of Theorem 1 and shows that, given a certain collection of sets of reserved colours on the vertices of G (for instance, the one provided by Lemma A), there exists a partial edge colouring that satisfies some desirable properties.

Lemma B. *Let $\{S_v : v \in V(G)\}$ be a collection of subsets of $[(1+\varepsilon)\Delta]$ that satisfies properties (A.1)–(A.3) from Lemma A. Then, there exists a proper partial acyclic edge colouring of G such that*

(B.1) every coloured edge $e = uv$ has a colour from $[(1+\varepsilon)\Delta] \setminus (S_u \cup S_v)$;
(B.2) for every vertex v and every colour $c \in S_v$ there are at most $\left(\frac{\varepsilon^2}{18}\right)^2 \frac{\Delta}{128}$ vertices $u \in N(v)$ satisfying that uv is uncoloured and $c \in S_u$, and
(B.3) every cycle C that can become completely bicoloured by using on the uncoloured edges $e = uv$ of C a colour in $S_u \cap S_v$, has at least 3 uncoloured edges.

The collection of sets provided by Lemma A can be used in Lemma B to obtain a partial acyclic edge colouring satisfying some nice properties. The following lemma shows that, using these properties, we can complete the partial acyclic edge colouring provided by Lemma B.

Lemma C. *Let $\gamma \in (0,1)$ be a given constant. Let χ be a partial acyclic edge colouring of G that satisfies the following properties,*

- *for every uncoloured edge e there is a list of colours of size exactly $\gamma\Delta$ that are not used by χ in the coloured edges adjacent to e,*
- *for every uncoloured edge e, every colour in its list appears in the list of at most $\gamma^2\Delta/128$ uncoloured edges adjacent to e,*
- *there are at most $(1+\varepsilon)\Delta$ colours in the union of all the lists, and*
- *every cycle that can be completely bicoloured in an extension of χ, has at least 3 uncoloured edges.*

Then, χ can be extended to an acyclic edge colouring of G.

Lemma A and Lemma C are relatively straightforward applications of the Local Lemma. Their proof is given in Section 4 and 6 respectively. We dedicate the rest of this section to give an outline of the proof of Lemma B.

We prove the existence of the desired colouring via an iterative procedure with i^* iterations, where i^* only depends on ε and will be defined later. For every $1 \le i \le i^*$, at the beginning of the i-th iteration, we will have a partial edge colouring of G. Moreover, for every uncoloured edge, we will also have a list of available colours for it. Then, we will perform the following steps in order to obtain a new partial edge colouring:

S.1: Each list of colours is truncated to the same size by removing some colours in an arbitrary way.
S.2: Each uncoloured edge is assigned a colour selected uniformly at random from its list.
S.3: Adjacent edges assigned the same colour are uncoloured.
S.4: For each edge with a newly assigned colour, an equalizing coin is flipped to decide if it gets uncoloured.
S.5: Colours retained on an edge are removed from the lists of the edges adjacent to it.
S.6: For every colour and vertex, an equalizing coin is flipped to decide if the colour is removed from all the lists of the edges incident to the vertex.

After performing these steps, we will show that there exist constants $L_{i+1}, T_{i+1}, R_{i+1}$ and Λ_{i+1} (to be set later) such that the partial edge colouring that we have obtained, satisfies the following properties:

– for every uncoloured edge, there are at least L_{i+1} choices of colours in its corresponding list,
– for every vertex v and every colour c, there are at most T_{i+1} edges $e = uv$ satisfying that e is uncoloured and c is in the list of e,
– for every vertex v and every colour c, there are at most R_{i+1} vertices $u \in N(v)$ satisfying that uv is uncoloured and $c \in S_u$, and
– every cycle that is partially bicoloured and that is *significant* (to be defined later) has at least Λ_{i+1} uncoloured *free* (to be defined later) edges.

Now, we define the quantities related to the partial edge colouring that we need to control throughout the iterative colouring. For every edge $e = uv$, let $L_1(e) := [(1+\varepsilon)\Delta] \setminus (S_u \cup S_v)$, be the initial list of available colours.

For every uncoloured edge e, let $L_i(e)$ be the list of available colours in e *before* the i-th iteration. (For the sake of clarity and with a slight abuse of notation, we also use $L_i(e)$ to refer to the list of available colours at edge e after the truncation at step S.1.) Let $\ell_i(e) := |L_i(e)|$.

For every vertex v and every colour c, let $T_i(v,c)$ be the set of edges e incident to v such that the following holds: *before* the i-th iteration e is uncoloured and $c \in L_i(e)$. Let $t_i(v,c) := |T_i(v,c)|$.

Finally, for every vertex v and every colour c, let $R_i(v,c)$ be the set of neighbours u of v such that the following holds: *before* the i-th iteration uv is uncoloured and $c \in S_u$. Let $r_i(v,c) := |R_i(v,c)|$.

Observe that $\ell_i(e)$, $t_i(v,c)$ and $r_i(v,c)$ control the first three quantities which we need to bound at every iteration. We define L_i, T_i and R_i recursively as follows. Let $L_1 := (1+\varepsilon/9)\,\Delta$, $T_1 := \Delta$ and $R_1 := \varepsilon\Delta/2$. Then, for every $i \geq 1$

$$\begin{aligned} L_{i+1} &:= (1-\mathrm{e}^{-2})^2\, L_i - L_i^{2/3}, \\ T_{i+1} &:= (1-\mathrm{e}^{-2})^2\, T_i + T_i^{2/3}, \\ R_{i+1} &:= (1-\mathrm{e}^{-2}) R_i + R_i^{2/3}. \end{aligned} \tag{1}$$

Let us now focus on the last property that the partial edge colouring must satisfy at the end of the i-th iteration. First of all, we give some definitions.

At the beginning of the i-th iteration, G has a partial edge colouring and each uncoloured edge $e = uv$ has two lists of colours: $L_i(e)$ (the colours that can be used for e during the i-th iteration) and $S_u \cap S_v$ (the colours that have been reserved for the end).

For every colour c and every uncoloured edge $e = uv$, we say that e is *c-reserved* if $c \in S_u \cap S_v$ and that e is *c-free* if $c \in L_i(e)$. Given a pair of colours $\{c,d\}$, we also use the term $\{c,d\}$-reserved to refer to edges that are either c-reserved, d-reserved, or both (and similarly for the free edges). By construction of $L_i(e)$, an edge cannot be both reserved and free. We say that an edge $e = uv$ is *c-compatible* if it has colour c or it is uncoloured and either c-reserved or c-free. We say that a cycle C is *$\{c,d\}$-compatible* if every edge in C is either c-compatible, d-compatible, or both.

Given a path $v_1, \ldots, v_\ell$ and an ordered pair of colours (c,d) we call the path (c,d)-alternating if, for every $j \geq 1$, $e = v_{2j-1}v_{2j}$ is a c-compatible edge and $e = v_{2j}v_{2j+1}$ is a d-compatible edge. For a $\{c,d\}$-compatible cycle C, we define its $\{c,d\}$-multiplicity as the smallest t such that C can be partitioned into t paths that are either (c,d)-alternating or (d,c)-alternating. We define the $\{c,d\}$-multiplicity of a path in the same way.

In order to control the multiplicity and the number of reserved edges in a cycle, we define

$$\Psi_i := 4^{2+i^*-i}.$$

We say that a cycle C is *$\{c,d\}$-significant* at the i-th iteration, if at the beginning of the i-th iteration, C is $\{c,d\}$-compatible, has $\{c,d\}$-multiplicity at most Ψ_i and contains at most Ψ_i $\{c,d\}$-reserved edges. (When the pair of colours $\{c,d\}$ is clear from the context, we simply refer to compatible and significant cycles, to multiplicity, and to reserved and free edges.)

Note that since Ψ_i is decreasing in i, the definition of significant cycles becomes stronger throughout the iterative procedure. Thus, the number of significant cycles that need to be controlled is considerably reduced at each iteration, which is a crucial point for our analysis.

Let $k := \lfloor g/2 \rfloor$, where g is the girth of G. For every pair of colours $\{c, d\}$ and every cycle C that is $\{c, d\}$-significant at the i-th iteration, let $\lambda_i^{\{c,d\}}(C)$ be its number of $\{c, d\}$-free edges at the beginning of the i-th iteration. In order to lower bound $\lambda_i^{\{c,d\}}(C)$, we define

$$\Lambda_i := \frac{2k}{2^{i-1}} - 4\Psi_1 i.$$

With this definition, we have $\Lambda_{i+1} = 2\Lambda_i + 4\Psi_1$.

Now, we are able to precisely state the properties of the partially edge-coloured graph that our colouring procedure must satisfy at every iteration. For every $0 \le i \le i^*$, we define:

Property P(i) . *With the definitions given above, the following is satisfied:*

(P.1) $\ell_{i+1}(e) \ge L_{i+1}$, *for every uncoloured edge e,*
(P.2) $t_{i+1}(v, c) \le T_{i+1}$, *for every vertex v and colour c,*
(P.3) $r_{i+1}(v, c) \le R_{i+1}$, *for every vertex v and colour c, and*
(P.4) $\lambda_{i+1}^{\{c,d\}}(C) \ge \Lambda_{i+1}$, *for every pair of colours* $\{c, d\}$ *and every cycle C that is* $\{c, d\}$-*significant at the* $(i + 1)$-*st iteration.*

Let us now define the total number of iterations $i^* = i^*(\varepsilon)$ as the smallest integer i such that

$$R_{i+1} < \left(\frac{\varepsilon^2}{18}\right)^2 \frac{\Delta}{128}. \tag{2}$$

We stress that if Δ is large enough, then i^* only depends on ε, since $R_1 = \varepsilon\Delta/2$ and $R_{i+1} = (1 - \mathrm{e}^{-2})R_i + R_i^{2/3}$. In particular, we have $i^* \le c \log_2(1/\varepsilon)$, for some constant $c > 0$.

Lemma 4. *If property $P(i^*)$ is satisfied, then there exists a proper partial acyclic edge colouring of G that satisfies the properties (B.1)–(B.3) of Lemma B.*

Proof. First of all, since every edge e that is coloured during the iterative procedure, is assigned a colour from $L_1(e) = [(1+\varepsilon)\Delta] \setminus (S_u \cup S_v)$, property (B.1) is satisfied. Moreover, by the definition of the stopping time i^* in (2), condition (B.2) is also satisfied.

Finally, we will show that $\lambda_{i^*+1}^{\{c,d\}}(C) \ge \Lambda_{i^*+1} = 2k/2^{i^*} - 4\Psi_1(i^* + 1)$ implies condition (B.3). Since i^* and Ψ_1 only depend on ε, we can consider $g = g(\varepsilon)$ (and thus, also k) large enough with respect to ε such that $\Lambda_{i^*+1} \ge 3$. This particularly implies that $k \approx 2^{i^*} = O\left(\varepsilon^{-c}\right)$. Let C be a partially edge-coloured cycle that can become completely bicoloured with colours c and d, by using in each uncoloured edge $e = uv$, colours from $S_u \cap S_v$. If C is significant at the $(i^* + 1)$-st iteration, then there are at least $\Lambda_{i^*+1} \ge 3$ free (and in particular, uncoloured) edges. Therefore assume that C is not significant. Since C can become bicoloured with colours c and d, it must be $\{c, d\}$-compatible and must have $\{c, d\}$-multiplicity 1. Thus, the only way that C fails to be significant is in the case that there are more than $\Psi_{i^*+1} = 4^{2+i^*-(i^*+1)} = 4$ reserved (and in particular, uncoloured) edges. In conclusion, any such cycle has at least 3 uncoloured edges and property (B.3) also holds. ∎

In order to prove that $P(i^*)$ holds, we first show that $P(0)$ holds by the hypothesis of Lemma B.

Lemma 5. *Property $P(0)$ is satisfied deterministically.*

Proof. For every edge e, we have $\ell_1(e) \geq (1+\varepsilon)\Delta - 2 \cdot \frac{4\varepsilon\Delta}{9} = L_1$ by the property (A.1). For every vertex v and every colour c, we have $t_1(v,c) \leq \Delta = T_1$. For every vertex v and every colour c, we have $r_1(v,c) \leq \frac{\varepsilon\Delta}{2} = R_1$ by the property (A.3). Finally, since for every pair of colours $\{c,d\}$, every $\{c,d\}$-significant cycle C contains at least $g \geq 2k$ uncoloured edges and at most Ψ_1 reserved edges (by the definition of significant cycles), C has at least $2k - \Psi_1 > 2k - 4\Psi_1 = \Lambda_1$ free edges. ∎

Thus, it suffices to show that for every $1 \leq i \leq i^*$, if property $P(i-1)$ holds, then with positive probability $P(i)$ also holds. In the remainder of this section, we provide an intuition on why this is true. We will do it by showing that the expected decrease of the quantities $\ell_i(e)$, $t_i(v,c)$ and $r_i(v,c)$, is governed by the parameters L_i, T_i and R_i, respectively. In Section 5 we present the details of it.

Observe that after the truncation at step S.1, we have $\ell_i(e) = L_i$ for every edge e. Assume that for every vertex v and every colour c, we have $t_i(v,c) = T_i$ and $r_i(v,c) = R_i$. This assumption will help us to derive an intuition on how the previous parameters drop at each iteration. The equalizing coins will be used to correct the possible fluctuations of the previous parameters. Since we are assuming that these parameters have a given value, it also makes sense to assume that the equalizing coins are not flipped (that is, steps S.4 and S.6 are skipped in this intuitive analysis).

For every vertex v and every colour c, let $\mathrm{Q}_i(v,c)$ be the probability that exactly one edge incident to v retains colour c assigned at step S.2. Observe that, by step S.5 in our procedure, this is precisely the same as the probability that c is removed from the lists of all the edges incident to v because an edge incident to v retains it. Since after step S.1 we have $\ell_i(e) = L_i$,

$$\mathrm{Q}_i(v,c) = \frac{T_i}{L_i}\left(1-\frac{1}{L_i}\right)^{T_i-1}\left(1-\frac{1}{L_i}\right)^{T_i-1} = \frac{T_i}{L_i}\left(1-\frac{1}{L_i}\right)^{2T_i-2} \approx \mathrm{e}^{-2},$$

where the last approximation follows since L_i and T_i are very close at the beginning of the iterative colouring (see definitions of L_1 and T_1) and they drop at the same speed, as it will be explained below. Since this probability does not depend on the choice of v and c, we write $\mathrm{Q}_i := \mathrm{Q}_i(v,c)$.

For $e = uv$, we can compute the expected value of $\ell_{i+1}(e)$,

$$\mathbb{E}\left[\ell_{i+1}(e)\right] = \mathbb{E}\left[\sum_{c \in L_i(e)} (1-\mathrm{Q}_i(u,c))(1-\mathrm{Q}_i(v,c))\right] = L_i(1-\mathrm{Q}_i)^2 \approx L_i(1-\mathrm{e}^{-2})^2 \approx L_{i+1},$$

where the last approximation follows from (1).

For every edge e and every colour c, let $\mathrm{P}_i(e,c)$ be the probability that every edge adjacent to e is not assigned colour c at step S.2. Then

$$\mathrm{P}_i(e,c) = \left(1-\frac{1}{L_i}\right)^{2(T_i-1)} \approx \mathrm{e}^{-2}.$$

Similarly as before, since the previous probability does not depend on the choice of e and c, we write $\mathrm{P}_i := \mathrm{P}_i(e,c)$. Observe that the probability that an edge e retains its colour is

exactly P_i: let c be the colour assigned to e at step S.2, then the probability it is retained at step S.3, is the probability c is not assigned to any of the edges adjacent to e.

For every vertex v and every colour c, there are two ways that an edge $e = vw \in T_i(v, c)$ satisfies $e \notin T_{i+1}(v, c)$: (1) either c is removed from the list corresponding to e at step S.5 because it was assigned to an edge incident to w (which happens with probability Q_i) or (2) e retains the colour assigned at step S.2 (which happens with probability P_i). Observe that if c is removed from the list corresponding to e because it was assigned to an edge incident to v, then, for every edge f incident to v we have $c \notin L_{i+1}(f)$ and we can stop tracking $T_{i+1}(v, c)$. Since the probability of (1) and (2) are essentially independent, it follows from the above estimations and by (1) that,

$$\mathbb{E}\left[t_{i+1}(v, c)\right] \approx T_i(1 - \mathrm{P}_i)(1 - \mathrm{Q}_i) \approx T_i(1 - \mathrm{e}^{-2})^2 \approx T_{i+1}.$$

Thus, as we stated before, roughly speaking, $\ell_i(e)$ and $t_i(v, c)$ drop at the same speed.

For every vertex v and every colour c, the only way that a vertex $u \in R_i(v, c)$ satisfies $u \notin R_{i+1}(v, c)$ is that the edge uv retains its assigned colour which occurs with probability P_i. Using (1) again,

$$\mathbb{E}\left[r_{i+1}(v, c)\right] = R_i(1 - \mathrm{P}_i) \approx R_i(1 - \mathrm{e}^{-2}) \approx R_{i+1}.$$

It is worth noticing that, on the one hand, the expected drop of $r_i(v, c)$ is smaller than the one of $\ell_i(e)$, but, on the other hand, $r_1(v, c) \leq R_1 = \varepsilon\Delta/2$ is much smaller than $\ell_1(e) \geq L_1 = (1 + \varepsilon/9)\Delta$.

The control of the number of free edges in significant cycles is more involved and will be done directly in Section 5.

4. PROOF OF LEMMA A

In this section we prove that there exists a collection of subsets of $[(1 + \varepsilon)\Delta]$ that satisfies some desirable properties.

Proof of Lemma A. For every vertex $v \in V(G)$ we construct the set S_v by selecting each colour c independently with probability $(1 + \varepsilon)^{-1/2}\varepsilon/3$. We show that with positive probability this collection of sets fulfils the conditions (A.1)–(A.3) in Lemma A.

Recall that for every edge $e = uv$, $S_e = S_v \cap S_u$. Let $S_{v,c} := \{u \in N(v) : c \in S_u\}$, be the set of neighbours of v that reserve colour c. Note that $S_{v,c} = R_1(v, c)$, with the definition of $R_i(v, c)$ given in Section 3. For every vertex v, edge e and colour c

$$\mathbb{E}\left[|S_v|\right] = \frac{\varepsilon(1 + \varepsilon)^{1/2}\Delta}{3}, \quad \mathbb{E}\left[|S_e|\right] = \frac{\varepsilon^2\Delta}{9} \quad \text{and} \quad \mathbb{E}\left[|S_{v,c}|\right] = \frac{\varepsilon\Delta}{3(1 + \varepsilon)^{1/2}}.$$

For every vertex v let A_v be the event that $|S_v| > \frac{4\varepsilon\Delta}{9}$, for every edge e let B_e be the event that $|S_e| < \frac{\varepsilon^2\Delta}{18}$ and for every vertex v and colour c let $C_{v,c}$ be the event that $|S_{v,c}| > \frac{\varepsilon\Delta}{2}$.

Observe that the random variables $|S_v|$, $|S_e|$ and $|S_{v,c}|$ are binomially distributed. We use Chernoff's inequality [16, chap. 5], which states that if X is a binomial with N trials and probability p, for any $\delta \in (0, 1)$:

$$\Pr\left(X \leq (1 - \delta)\mathbb{E}\left[X\right]\right) < e^{-\frac{\delta^2}{2}\mathbb{E}[X]},$$

and

$$\Pr\left(X \geq (1+\delta)\mathbb{E}\left[X\right]\right) < e^{-\frac{\delta^2}{3}\mathbb{E}[X]}.$$

Recall that $\varepsilon \leq \varepsilon_0 = 10^{-3}$. Then, we have

$$\Pr\left(A_v\right) \leq \Pr\left(|S_v| > (1+1/4)\mathbb{E}\left[|S_v|\right]\right) \leq \mathrm{e}^{-(1/4)^2\,\mathbb{E}[|S_v|]/3} = \mathrm{e}^{-\Omega(\Delta)}$$

$$\Pr\left(B_e\right) = \Pr\left(|S_e| < (1-1/2)\mathbb{E}\left[|S_e|\right]\right) \leq \mathrm{e}^{-(1/2)^2\,\mathbb{E}[|S_e|]/2} = \mathrm{e}^{-\Omega(\Delta)}$$

$$\Pr\left(C_{v,c}\right) \leq \Pr\left(|S_{v,c}| > (1+1/2)\mathbb{E}\left[|S_{v,c}|\right]\right) \leq \mathrm{e}^{-(1/2)^2\,\mathbb{E}\left[|S_{v,c}|\right]/3} = \mathrm{e}^{-\Omega(\Delta)}.$$

Observe that all the events A_v are mutually independent. An event A_v depends on at most Δ events of type B, precisely the ones for which $e = uv$, and on at most $(1+\varepsilon)\Delta^2$ of type C, one for each vertex $u \in N(v)$ and colour c. Analogously, an event B_e depends on at most 2 events of type A, on at most $2(\Delta-1)$ other events of type B, and on at most $2(1+\varepsilon)\Delta^2$ events of type C. Finally each event $C_{v,c}$ depends on at most Δ events of type A, on at most Δ^2 events of type B and on at most Δ^2 other events of type C.

The Lovász Local Lemma (in the form of [16, p. 221]) directly shows that if Δ is large enough with respect to ε, with positive probability all the events do not hold at the same time, thus proving the lemma. ∎

5. PROOF OF LEMMA B

In this section we prove the lemma where most of the acyclic edge colouring is constructed. First we compute the precise expectation of each random variable involved in the iterative colouring procedure. Then, we show that these random variables are concentrated. Finally we prove that with positive probability at the end of each iteration the graph has a partial edge colouring that satisfies the desired properties.

5.1. The Expected Values

Recall all the definitions given in Section 3. In this subsection we compute the expected values of $\ell_{i+1}(e)$, $t_{i+1}(v,c)$ and $r_{i+1}(v,c)$, given that property $P(i-1)$ holds. For simplicity, let $\eta := (1-\mathrm{e}^{-2})^2$ throughout this section.

The first lemma in this section shows that L_i, T_i, and R_i do not become too small before the i^*-th iteration.

Lemma 6. *There exist positive constants ε_1, ε_2, and ε_3 (only depending on ε) such that for every $1 \leq i \leq i^*+1$,*

$$L_i \geq \varepsilon_1\Delta, \qquad T_i \geq \varepsilon_2\Delta, \qquad R_i \geq \varepsilon_3\Delta.$$

Proof. Recall that i^* is defined in (2) as the smallest integer such that $R_{i+1} < \left(\frac{\varepsilon^2}{18}\right)^2 \frac{\Delta}{128} = \frac{\varepsilon^3}{18^2}\cdot\frac{R_1}{128}$. Besides, by (1) we also have $R_{i^*+1} \geq \eta^{i^*/2}R_1$. Therefore, $\eta^{i^*} \geq \left(\frac{\varepsilon^3}{18^2\cdot 128}\right)^2$. Let $\varepsilon_1 = \varepsilon^8$. For every $i \leq i^*+1$, it follows that

$$L_i \geq \eta^{i^*}L_1 - i^*L_1^{2/3} \geq (1-o(1))\left(\frac{\varepsilon^3}{18^2\cdot 128}\right)^2\left(1+\frac{\varepsilon}{9}\right)\Delta \geq \varepsilon_1\Delta,$$

where we used that $\varepsilon \leq \varepsilon_0 = 10^{-3}$. Similar arguments can be applied for T_i and R_i. ∎

Recall the definitions of L_1, T_1 and R_1 and define the following three sequences for every $i \geq 0$,

$$L'_{i+1} := \eta^i L_1, \qquad T'_{i+1} := \eta^i T_1, \qquad R'_{i+1} := \eta^{i/2} R_1.$$

The next lemma shows that L_i, T_i and R_i, as defined in (1), are very close to L'_i, T'_i and R'_i respectively.

Lemma 7. *For every* $1 \leq i \leq i^* + 1$, *we have*

(a) $|L_i - L'_i| \leq (L'_i)^{5/6} = o(L'_i)$,
(b) $|T_i - T'_i| \leq (T'_i)^{5/6} = o(T'_i)$,
(c) $|R_i - R'_i| \leq (R'_i)^{5/6} = o(R'_i)$.

Proof. Let us prove part (a) using induction on i. The proofs of parts (b) and (c) use identical arguments and are omitted.

The base case $i = 1$ is clearly true as $L'_1 = L_1$. If (a) holds for i, then

$$|L'_{i+1} - L_{i+1}| = L'_{i+1} - L_{i+1} = \eta L'_i - \eta L_i + L_i^{2/3} \leq \eta (L'_i)^{5/6} + L_i^{2/3},$$

where the first equality comes from the obvious fact that $L_j \leq L'_j$ for all j. To see that the expression above is at most $(L'_{i+1})^{5/6}$, it suffices to note that

$$(L'_{i+1})^{5/6} - \eta (L'_i)^{5/6} = (\eta^{5/6} - \eta)(L'_i)^{5/6} \geq (L'_i)^{2/3} \geq L_i^{2/3},$$

where in the first inequality, we used that $L'_i \geq L_i \geq \varepsilon_1 \Delta$ from Lemma 6. ∎

This other lemma is a consequence of Lemma 7 and shows that L_i and T_i are close throughout the iterative colouring.

Lemma 8. *For every* $1 \leq i \leq i^* + 1$, *we have* $L_i/T_i = 1 + \varepsilon/9 + o(1)$.

Proof. Since $L_i \leq L'_i$ and $T_i \geq T'_i$,

$$\frac{L_i}{T_i} \leq \frac{L'_i}{T'_i} = \frac{L_1 \eta^{i-1}}{T_1 \eta^{i-1}} = 1 + \frac{\varepsilon}{9}.$$

The other direction follows from Lemma 7,

$$\frac{L_i}{T_i} \geq \frac{(1 - o(1))L'_i}{(1 + o(1))T'_i} = 1 + \frac{\varepsilon}{9} - o(1).$$

∎

Recall that $L_i(e)$ is the list of colours that are still available to e at the beginning of the i-th iteration. For a vertex v, we define

$$L_i(v) := \bigcup_{\substack{e \ni v: \\ e \text{ uncoloured}}} L_i(e),$$

to be the union of the colour lists of all uncoloured edges incident to v. When we say that a colour c is removed from the list $L_i(v)$, it is also removed from every list $L_i(e)$ with e uncoloured and incident to v.

In Section 3 we sketched the steps that we perform at the i-th iteration of the procedure. Here we make it precise, provided that $P(i-1)$ is satisfied:

S.1: For every uncoloured edge $e = uv$, truncate $L_i(e)$ by removing colours in an arbitrary way until it has size precisely L_i (recall that by (P.1), we have $\ell_i(e) \geq L_i$). When a colour c is removed from $L_i(e)$, e is also removed from $T_i(v,c)$ and $T_i(u,c)$.
S.2: For every uncoloured edge e, assign it a colour chosen uniformly at random from $L_i(e)$.
S.3: Uncolour every edge e which is assigned the same colour as one of its adjacent edges.
S.4: If e is assigned a colour c and is not uncoloured in the previous step, uncolour it with probability $\mathrm{Eq}_i(e,c)$ (to be defined below).
S.5: For each vertex v and colour $c \in L_i(v)$, if c is retained by an edge in $T_i(v,c)$, then remove c from $L_i(v)$.
S.6: For each vertex v and colour $c \in L_i(v)$, if c is not retained by an edge in $T_i(v,c)$, then remove c from $L_i(v)$ with probability $\mathrm{Vq}_i(v,c)$ (to be defined below).

As we pointed out before, the steps S.4 and S.6 correspond to the equalizing coin flips and are performed to correct the fluctuations of the process.

For an edge $e = uv$ and a colour $c \in L_i(e)$, let $\mathrm{P}_i(e,c)$ denote the probability that no edge adjacent to e is assigned colour c. It follows from Lemma 6 and Lemma 8 that

$$\begin{aligned}
\mathrm{P}_i(e,c) &= \left(1 - \frac{1}{L_i}\right)^{t_i(u,c)+t_i(v,c)-2} \\
&> \exp\left(-\frac{t_i(u,c)+t_i(v,c)-2}{L_i}\right) - O\left(\frac{1}{L_i}\right) \\
&\geq \exp\left(-\frac{2T_i-2}{L_i}\right) - O\left(\frac{1}{L_i}\right) \\
&\geq \mathrm{e}^{-2}\left(\exp\left\{\frac{2\varepsilon}{9+\varepsilon} - o(1) + \frac{2}{L_i}\right\}\right) - O\left(\frac{1}{L_i}\right) \\
&> \mathrm{e}^{-2}.
\end{aligned}$$

The last inequality holds since Δ is large with respect to ε and Lemma 6 shows that $L_i \geq \varepsilon_1 \Delta$.

Hence, one can choose $\mathrm{Eq}_i(e,c) := 1 - 1/(\mathrm{e}^2 \mathrm{P}_i(e,c)) > 0$. This ensures that the probability that e retains c, conditional on c being assigned to e, is precisely $(1-\mathrm{Eq}_i(e,c))\mathrm{P}_i(e,c) = \mathrm{e}^{-2}$. Therefore, the expected value of $r_{i+1}(v,c)$, is precisely $(1-\mathrm{e}^{-2})r_i(v,c)$.

Similarly, for every colour $c \in L_i(v)$, the probability that the colour c is not retained by an edge incident to v is $\mathrm{Q}_i(v,c) = 1 - t_i(v,c)/(\mathrm{e}^2 L_i) > 1 - \mathrm{e}^{-2}$. Thus choosing $\mathrm{Vq}_i(v,c) := 1 - (1-\mathrm{e}^{-2})/\mathrm{Q}_i(v,c)$ ensures that the probability that c remains in $L_i(v)$ is precisely $(1-\mathrm{Vq}_i(v,c))\mathrm{Q}_i(v,c) = 1 - \mathrm{e}^{-2}$.

Consider an edge $e = uv$. For a colour $c \in L_i(e)$ to be in $L_{i+1}(e)$, it must be in both $L_{i+1}(u)$ and $L_{i+1}(v)$. If the two events $\{c \in L_{i+1}(u)\}$ and $\{c \in L_{i+1}(v)\}$ were independent, then the expected value of $\ell_{i+1}(v,c)$ would be precisely $(1-\mathrm{e}^{-2})^2\,\ell_i(e)$. It can be shown that this estimation is not far from being accurate.

For an edge $e \in T_i(v,c)$, in order that e also belongs to $T_{i+1}(v,c)$, e must stay uncoloured and c must also be in $L_{i+1}(e)$. If the two events $\{e \text{ is uncoloured}\}$ and $\{c \in L_{i+1}(e)\}$ were independent then the expected value of $t_{i+1}(v,c)$ would be precisely $(1-\mathrm{e}^{-2})^2\,t_i(v,c)$. Although

the assumption of independence is not true, it turns out that the effect of dependency is small.

The following lemma turns the above intuitions into precise statements. The proofs are omitted as they are exactly the same as those in [16, chap 14.3].

Lemma 9. *Suppose that $P(i-1)$ holds. Then, for every vertex v, every edge e and every colour $c \in L_i(v)$,*

(a) $\mathbb{E}[r_{i+1}(v,c)] = \left(1-\mathrm{e}^{-2}\right) r_i(v,c)$,
(b) $\mathbb{E}[\ell_{i+1}(e)] \geq (1-\mathrm{e}^{-2})^2 L_i$, *and*
(c) $\mathbb{E}[t_{i+1}(v,c)] \leq (1-\mathrm{e}^{-2})^2 T_i + 1$.

5.2. The Concentration

This subsection sketches the proof that $\ell_{i+1}(e)$, $t_{i+1}(v,c)$ and $r_{i+1}(v,c)$ have highly concentrated distributions given that $P(i-1)$ holds.

Lemma 10. *There exists a constant $\beta > 0$ such that for every constant $\gamma > 0$ and for every $1 \leq i \leq i^*$:*

(a) For every vertex v and every colour $c \in L_i(v)$,

$$\Pr\left(|r_{i+1}(v,c) - \mathbb{E}[r_{i+1}(v,c)]| \geq \gamma\sqrt{\mathbb{E}[r_{i+1}(v,c)]}\Delta^{1/6}\right) \leq \exp\{-\beta\gamma^2\Delta^{1/3}\};$$

(b) For every edge e,

$$\Pr\left(|\ell_{i+1}(e) - \mathbb{E}[\ell_{i+1}(e)]| > \gamma\sqrt{L_{i+1}}\Delta^{1/6}\right) \leq \exp\{-\beta\gamma^2\Delta^{1/3}\};$$

(c) For every vertex v and colour c,

$$\Pr\left(|t_{i+1}(v,c) - \mathbb{E}[t_{i+1}(v,c)]| > \gamma\sqrt{L_{i+1}}\Delta^{1/6}\right) \leq \exp\{-\beta\gamma^2\Delta^{1/3}\}.$$

The tool used to prove the above lemma is Talagrand's inequality [16, chap. 10]. It implies that to show a random variable X is concentrated, it suffices to verify that there exist two constants ξ_1 and ξ_2 such that:

(a) changing the outcome of one random choice can affect X by at most ξ_1;
(b) for every s, if $X \geq s$, then there exists a set of at most $\xi_2 s$ random choices whose outcomes certify that $X \geq s$.

If these two conditions hold, then

$$\Pr\left(|X - \mathbb{E}[X]| > \alpha\sqrt{\mathbb{E}[X]}\right) \leq \exp\{-\beta\alpha^2\},$$

for $\beta < \frac{1}{10\xi_1^2\xi_2}$ and $\alpha \leq \sqrt{\mathbb{E}[X]}$. In the proof of the Lemma 10, the random choices are the colour assignments and equalizing coin flips. We can take β as the minimum of the β's for the three types of random variables and $\alpha = \gamma\Delta^{1/6}$.

For the part (a) of Lemma 10, note that changing the colour assigned to an edge from c_1 to c_2 can only uncolour at most one edge incident to v, because two or more incident edges assigned colour c_2 would be uncoloured anyway. Meanwhile changing the outcome of one equalizing coin flip can only change $r_{i+1}(v,c)$ by at most one. And for s edges in $R_i(v,c)$ which get uncoloured, either by conflicts or by equalizing coin flips, at most s outcomes of random choices are enough to certify it.

The proofs of parts (b) and (c) are more complicated. The idea is to split the random variables into a linear combination of related random variables, which can be shown to have concentrated distribution by Talagrand's inequality. The detailed proofs are given in [16, chap. 14.4].

5.3. Control at Each Iteration

In this section we will show that the property $P(i)$ defined in Section 3 holds for every $0 \le i \le i^*$ with positive probability. We define the following set of events,

$$\begin{aligned}
A_C^{\{c,d\}} &:= [\lambda_{i+1}(C) < \Lambda_{i+1}] \\
&\quad \cap [C \text{ is } \{c,d\}\text{-significant} \\
&\quad \text{at the } (i+1)\text{-st iteration}] && \text{for every pair of colours } c,d \text{ and every cycle } C \text{ which is } \{c,d\}\text{-significant at the } i\text{-th iteration,} \\
B_e &:= [\ell_{i+1}(e) < L_{i+1}] && \text{for every edge } e \text{ uncoloured,} \\
C_{v,c} &:= [t_{i+1}(v,c) > T_{i+1}] && \text{for every vertex } v \text{ and every colour } c\text{, and} \\
D_{v,c} &:= [r_{i+1}(v,c) > R_{i+1}] && \text{for every vertex } v \text{ and every colour } c.
\end{aligned}$$

Let $\mathcal{E}$ be the collection of the above events. Observe that $\Pr(P(i)) = \Pr\left(\cap_{E\in\mathcal{E}}\overline{E}\right)$ so it suffices to lower bound the latter probability.

The next lemma bounds the probability of the events B_e, $C_{v,c}$ and $D_{v,c}$ using the results obtained in Sections 5.1 and 5.2.

Lemma 11. *Given that $P(i-1)$ holds, for every uncoloured edge e, every vertex v, and colour c*

(a) $\Pr(B_e) = \mathrm{e}^{-\Omega(\Delta^{1/3})}$;
(b) $\Pr\left(C_{v,c}\right) = \mathrm{e}^{-\Omega(\Delta^{1/3})}$;
(c) $\Pr\left(D_{v,c}\right) = \mathrm{e}^{-\Omega(\Delta^{1/3})}$.

Proof. For part (*a*), recall that $L_{i+1} = (1-\mathrm{e}^{-2})^2 L_i - L_i^{2/3}$ by definition. By Lemma 6, $L_i \ge \varepsilon_1\Delta$ and thus, there exists a constant γ such that $L_i^{2/3} \ge \gamma\sqrt{L_{i+1}}\Delta^{\frac{1}{6}}$. Combining these results with the computation of $\mathbb{E}[\ell_{i+1}(e)]$ in Lemma 9

$$\begin{aligned}
\Pr(B_e) &= \Pr(\ell_{i+1}(e) < L_{i+1}) \\
&= \Pr\left(\ell_{i+1}(e) < (1-\mathrm{e}^{-2})^2 L_i - L_i^{\frac{2}{3}}\right) \\
&\le \Pr\left(\ell_{i+1}(e) < \mathbb{E}[\ell_{i+1}(e)] - \gamma\sqrt{L_{i+1}}\Delta^{\frac{1}{6}}\right) \\
&\le \exp\{-\beta\gamma^2\Delta^{\frac{1}{3}}\},
\end{aligned}$$

where the last inequality follows from Lemma 10.

The proofs of the parts (b) and (c) are along the same lines of the previous one and use the bounds on the expected values derived in Lemma 9 and the concentration inequalities provided in Lemma 10. ∎

Now we compute a bound on the probability of $A_C^{\{c,d\}}$.

Lemma 12. *For every $0 \le i \le i^*$, every pair of colours $\{c,d\}$ and every cycle C that is $\{c,d\}$-significant at the i-th iteration, we have*

$$\Pr\left(A_C^{\{c,d\}}\right) \le \Lambda_{i+1}\lambda_i(C)^{2\Psi_1+\Lambda_{i+1}} L_i^{-(\lambda_i(C)-\Lambda_i/2+4\Psi_1)}.$$

Moreover, we also have

$$-\log_\Delta \Pr\left(A_C^{c,d}\right) \ge \lambda_i(C)/3.$$

Proof. Recall that a cycle C is $\{c,d\}$-significant at the i-th iteration, if at the beginning of the i-th iteration, C is $\{c,d\}$-compatible, has $\{c,d\}$-multiplicity at most Ψ_i and contains at most Ψ_i $\{c,d\}$-reserved edges. If a cycle C is $\{c,d\}$-significant at the $(i+1)$-st iteration, it must be $\{c,d\}$-significant at the i-th iteration. Thus by property $P(i-1)$, C contains at least Λ_i free edges at the beginning of the i-th iteration; that is, $\lambda_i(C) \ge \Lambda_i$. Moreover, C contains at most Ψ_i reserved edges. For C to stay significant, after the i-th iteration C must have multiplicity at most Ψ_{i+1}. This implies that the cycle C can be partitioned into at most Ψ_{i+1} paths that are either (c,d)-alternating or (d,c)-alternating. Thus there are at most $\lambda_i(C)^{\Psi_{i+1}}$ ways to partition the free edges into Ψ_{i+1} groups according to which alternating path they belong to. Once this partition is decided, for each such group one should choose if they form a (c,d)-alternating path or a (d,c)-alternating. Using that $\lambda_i(C) \ge 2$ and that $\Psi_{i+1} \le \Psi_1$, it follows that there are at most $(2\lambda_i(C))^{\Psi_{i+1}} \le \lambda_i(C)^{2\Psi_1}$ ways to decide which edge of C should have colour c or d such that $A_C^{\{c,d\}}$ holds.

Another condition for C to keep being significant is that no reserved edge in C gets coloured. As only an upper bound of $\Pr\left(A_C^{\{c,d\}}\right)$ is needed, we can assume this condition is satisfied.

Thus we can upper bound the probability of $A_C^{\{c,d\}}$ using a union bound over all the possible sets of free edges of size at most Λ_{i+1} and over the at most $\lambda_i(C)^{2\Psi_1}$ possible ways to partition the $\lambda_i(C)$ free edges into alternating paths. Recall that at the beginning of the i-th iteration the cycle C has $\lambda_i(C)$ free edges, and that the list of each uncoloured edge has size exactly L_i. Also recall that $\Lambda_i := 2k/2^{i-1} - 4\Psi_1 i$ and that $\Psi_i := 4^{2+i^*-i}$. We have

$$\begin{aligned}
\Pr\left(A_C^{\{c,d\}}\right) &\le \lambda_i(C)^{2\Psi_1} \sum_{t=0}^{\Lambda_{i+1}} \binom{\lambda_i(C)}{t} L_i^{-(\lambda_i(C)-t)} \\
&\le \Lambda_{i+1}\lambda_i(C)^{2\Psi_1+\Lambda_{i+1}} L_i^{-(\lambda_i(C)-\Lambda_{i+1})} \\
&= \Lambda_{i+1}\lambda_i(C)^{2\Psi_1+\Lambda_{i+1}} L_i^{-(\lambda_i(C)-\Lambda_i/2)-(\Lambda_i/2-\Lambda_{i+1})} \\
&\le \Lambda_{i+1}\lambda_i(C)^{2\Psi_1+\Lambda_{i+1}} L_i^{-(\lambda_i(C)-\Lambda_i/2)-\Psi_1(2i+4)} \\
&\le \Lambda_{i+1}\lambda_i(C)^{2\Psi_1+\Lambda_{i+1}} L_i^{-(\lambda_i(C)-\Lambda_i/2+4\Psi_1)}.
\end{aligned}$$

This proves the first part of the lemma. For the second part, using $\sum_{i=0}^{b}\binom{a}{i} \le 2^a$, we obtain,

$$\Pr\left(A_C^{\{c,d\}}\right) \le \lambda_i(C)^{2\Psi_1} 2^{\lambda_i(C)} L_i^{-(\lambda_i(C)-\Lambda_i/2)} \le \lambda_i(C)^{2\Psi_1} 2^{\lambda_i(C)} L_i^{-(\lambda_i(C)/2)},$$

where we used that by induction hypothesis $P(i-1)$ holds, and thus $\lambda_i(C) \ge \Lambda_i$.

Recall that $L_i \geq \varepsilon_1 \Delta$ by Lemma 6. Hence,

$$\begin{aligned} -\log_\Delta \Pr\left(A_C^{c,d}\right) &\geq \frac{\lambda_i(C)}{2} \cdot \left(\log_\Delta L_i - 2\log_\Delta 2\right) - 2\Psi_1 \log_\Delta \lambda_i(C) \\ &\geq \frac{\lambda_i(C)}{2} \cdot \left(1 + \log_\Delta (\varepsilon_1/4)\right) - 2\Psi_1 \log_\Delta \lambda_i(C) \\ &\geq \frac{\lambda_i(C)}{3}, \end{aligned}$$

provided that Δ and g (and thus, k) are large enough with respect to ε. ∎

We will use the following modified version of the Weighted Lovász Local Lemma (see, e.g., [16, p. 221]) to lower bound $\Pr\left(\cap_{E\in\mathcal{E}}\overline{E}\right)$. Its proof can be easily derived from the general version of the local lemma (see, e.g., [16, p. 222]), and we include it here for the sake of completeness:

Lemma 13 (General Lovász Local Lemma). *Consider a set $\mathcal{E} = \{E_1, \ldots, E_M\}$ of events such that each E_r is mutually independent of $\mathcal{E} \setminus (\mathcal{D}_r \cup \{E_r\})$, for some $\mathcal{D}_r \subseteq \mathcal{E}$. If there exist constants $x_1, \ldots, x_M \in [0, 1)$ such that for each $1 \leq r \leq M$*

$$\Pr(E_r) \leq x_r \prod_{E_s \in \mathcal{D}_r} (1 - x_s),$$

then with positive probability, none of the events in $\mathcal{E}$ occur.

Lemma 14 (Alternative version of the Weighted Lovász Local Lemma). *Let $\beta > 0$ be a constant and let $\nu > 0$ be such that $\beta = \mathrm{e}^{2\log 2/\nu}$. Consider a set $\mathcal{E} = \{E_1, \ldots, E_M\}$ of events such that each E_r is mutually independent of $\mathcal{E} \setminus (\mathcal{D}_r \cup \{E_r\})$, for some $\mathcal{D}_r \subseteq \mathcal{E}$. If there exist constants $w_1, \ldots, w_M \geq 1$ and $h_1, \ldots, h_M$ such that*

(a) $\Pr(E_r) \leq h_r$;
(b) $\sum_{E_s \in \mathcal{D}_r} \beta^{w_s} h_s \leq w_r/\nu$;
(c) $\beta^{w_r} h_r \leq 1/2$;

then with positive probability, none of the events in $\mathcal{E}$ occur.

Proof. Let $x_r := \beta^{w_r} h_r$. By assumption $x_r \leq 1/2$. Thus $1 - x_r \geq \mathrm{e}^{-\alpha x_r}$, where $\alpha = 2\log 2$. Observe also that $\beta = \mathrm{e}^{\alpha/\nu}$. Therefore

$$\begin{aligned} x_r \prod_{E_s \in \mathcal{D}_r} (1 - x_s) &\geq x_r \exp\left\{-\alpha \sum_{E_s \in \mathcal{D}_r} x_s\right\} \\ &= \beta^{w_r} h_r \exp\left\{-\alpha \sum_{E_s \in \mathcal{D}_r} \beta^{w_s} h_s\right\} \\ &\geq \beta^{w_r} h_r \exp\left\{-\frac{\alpha w_r}{\nu}\right\} \\ &= h_r \geq \Pr(E_r). \end{aligned}$$

Thus the lemma follows from the General Lovász Local Lemma. ∎

For every $\lambda \geq 0$, every edge e and every pair of colours $\{c, d\}$, let $\mathcal{C}^{\{c,d\}}(e, \lambda)$ be the set of $\{c, d\}$-significant cycles at the i-th iteration that contain the edge e, that contain exactly λ free edges, and that can possibly remain significant at the $(i+1)$-st iteration. Let $\mathcal{C}^{\{c,d\}}(e) = \cup_{\lambda \geq 0} \mathcal{C}^{\{c,d\}}(e, \lambda)$. The following lemma controls the contribution of the probability of the events $A_C^{\{c,d\}}$ for all $C \in \mathcal{C}^{\{c,d\}}(e)$.

Lemma 15. *Suppose that $P(i-1)$ holds. Then, for every pair of colours $\{c, d\}$ and every free c-compatible edge e,*

$$\sum_{C \in \mathcal{C}^{\{c,d\}}(e)} \left(1 + \frac{\varepsilon}{10}\right)^{\lambda_i(C)/2} \Pr\left(A_C^{\{c,d\}}\right) = O(\Delta^{-7}). \tag{3}$$

Proof. We count the number of cycles that could possibly remain $\{c, d\}$-significant after the i-th iteration and that contain a given free edge $e = uv$, by constructing them from the edge e. At each step of the construction, we consider how many of the unused edges can extend the current cycle. We will use the fact that, since $P(i-1)$ holds, every such cycle has many free edges.

Consider the graph G' obtained from G by keeping all the edges that participate in cycles that could be $\{c, d\}$-significant at the $(i+1)$-st iteration. In particular, all these edges are $\{c, d\}$-compatible. We are going to construct a subgraph $H \subseteq G'$ rooted at v. Let $V_0 := \{v\}$ and let $H_0 := G'[V_0]$ be the empty subgraph formed by v. We construct $H_{j+1} := G'[V_{j+1}]$, with $V_{j+1} := V_j \cup W_{j+1}$, where W_{j+1} is the set of vertices $w \in V(G) \setminus V_j$ that satisfy

(a) w is at distance 1 from W_j in $G' \setminus e$, and
(b) there exists a path in $G' \setminus e$ connecting v to w using exactly one vertex in each of $W_0, \ldots, W_j$ that contains:
   (b.1) at most Λ_{i+1} free edges,
   (b.2) has $\{c, d\}$-multiplicity at most Ψ_{i+1}, and
   (b.3) at most Ψ_{i+1} $\{c, d\}$-reserved edges.

Let τ be the smallest integer such that $W_{\tau+1} = \emptyset$ and consider $H := G'[V_\tau]$.

Claim . For every $0 \leq j < \tau$, each vertex in W_{j+1} has a unique neighbour in H_{j+1} which belongs to W_j. In particular, the subgraph H is a rooted tree.

Proof. Equivalently, we want to show that for every vertex $w \in H$ there is a unique path in H to the root v. For the sake of contradiction, suppose that there exists a cycle in H. Let j be the smallest integer such that H_j contains at least one cycle.

Adding W_j can only create a cycle in one of two ways: either (1) there exists $w \in W_j$ such that w is connected two vertices in H_{j-1}, or (2) there exist $w_1, w_2 \in W_j$ such that there is a path from w_1 to w_2 in $G'[W_j]$. If (1) occurs, we can find a cycle D in H_j that consists of two internally-disjoint paths p_1 and p_2 contained in H_{j-1}, and the vertex w. If (1) does not occur, for every $w \in W_j$, there exists a unique path in H_{j-1} that connects the unique neighbour of w in W_{j-1} to v. Therefore, we can assume that $w_1 w_2$ is an edge of G' (otherwise take any of the edges in the path between w_1 and w_2 in $G'[W_j]$) and we obtain a cycle D formed by two internally-disjoint paths p_1 and p_2 contained in H_{j-1} and the vertices w_1 and w_2.

By our construction, p_1 and p_2 have at most one vertex in each set W_i, for $i < j$, and thus, each of them has fewer than Λ_{i+1} free edges, multiplicity at most Ψ_{i+1} and at most

Ψ_{i+1} reserved edges. Since D is composed of p_1, p_2 and at most three extra edges (the one connecting p_i to w_i, $i \in \{1,2\}$ and w_1w_2), it must have at most $2\Lambda_{i+1} + 3 \leq \Lambda_i$ free edges, multiplicity at most $2\Psi_{i+1} + 3 \leq \Psi_i$ and at most $2\Psi_{i+1} + 3 \leq \Psi_i$ reserved edges. Thus D was $\{c,d\}$-significant at the i-th iteration but contains fewer than Λ_i free edges, thus giving a contradiction with the fact that every $\{c,d\}$-significant cycle has at least Λ_i free edges, since $P(i-1)$ is satisfied. Hence, there exists a unique path in H from any vertex w to the root v. This implies that every vertex in W_{j+1} has a unique neighbour in H_{j+1} (which belongs to W_j) and that H is acyclic. ∎

Recall that for a cycle C to be able to be $\{c,d\}$-significant at the $(i+1)$-st iteration, that is $C \in \mathcal{C}^{\{c,d\}}(e)$, it must satisfy the following three properties:

- C is $\{c,d\}$-significant at the i-th iteration,
- C has $\{c,d\}$-multiplicity at most Ψ_{i+1}: during an iteration the multiplicity of a cycle can only increase, and
- C contains at most Ψ_{i+1} $\{c,d\}$-reserved edges: the number of reserved edges only decreases when one of them retains a colour different than c or d, and in such case, the edge is neither c- nor d-compatible, and the C is no longer $\{c,d\}$-significant.

By construction of G', if $C \in \mathcal{C}^{\{c,d\}}(e)$, then C is contained in G'. Since C contains the edge e and since H is acyclic, there exists a simple path $P \subset C$ in H from v to a leaf w' of H such that P intersects each set W_i exactly once. (Observe that C might intersect H in a very complicated way, but the path P has a simple structure.) As we explained before, if $C \in \mathcal{C}^{\{c,d\}}(e)$, then it has $\{c,d\}$-multiplicity at most Ψ_{i+1} and contains at most Ψ_{i+1} $\{c,d\}$-reserved edges. Fix C and create P at the same time we construct H. Since conditions (b.2) and (b.3) are always satisfied by the properties of C, during the construction of H the path P can only stop growing because condition (b.1) is violated, which implies that P contains exactly Λ_{i+1} free edges. (The endpoint w' of P must be a leaf in H since no neighbour of w' can be added to H any more.) Thus, $C \setminus P$ must be a path that contains exactly $\lambda_i(C) - \Lambda_{i+1}$ free edges, has multiplicity at most Ψ_{i+1} and contains at most Ψ_{i+1} reserved edges.

Let $P_u(\lambda_f, \lambda_r, \lambda_s)$ be the set of paths starting from u and containing λ_f free edges, λ_r reserved edges and multiplicity λ_s. There are $\binom{\lambda_f+\lambda_r}{\lambda_r} \leq (\lambda_f + \lambda_r)^{\lambda_r}$ ways to select which edges are reserved and which ones are free in the path. Once this is fixed, there are at most $(2(\lambda_f + \lambda_r))^{\lambda_s}$ ways to partition free and reserved edges into groups according to which alternating path they belong to. Given the previous selections, there are at most $T_i^{\lambda_f}\Delta^{\lambda_r}$ paths starting from u that agree with them: a vertex can be incident either to one coloured edge, to at most T_i free edges or to at most Δ reserved edges (the colour corresponding to these edges is either c or d depending on their distance to e). Therefore, summing over all the possible selections, we have $|P_u(\lambda_f, \lambda_r, \lambda_s)| \leq 2^{\lambda_s}(\lambda_f + \lambda_r)^{\lambda_r+\lambda_s}T_i^{\lambda_f}\Delta^{\lambda_r}$.

Thus, the number of candidate paths P that are induced by some cycle $C \in \mathcal{C}^{\{c,d\}}(e, \lambda)$ is at most

$$\begin{aligned}\sum_{\lambda_r=0}^{\Psi_{i+1}}\sum_{\lambda_s=0}^{\Psi_{i+1}} |P_u(\lambda - \Lambda_{i+1}, \lambda_r, \lambda_s)| &\leq \sum_{\lambda_r=0}^{\Psi_{i+1}}\sum_{\lambda_s=0}^{\Psi_{i+1}} 2^{\lambda_s}(\lambda - \Lambda_{i+1} + \lambda_r)^{\lambda_r+\lambda_s} T_i^{\lambda-\Lambda_{i+1}}\Delta^{\lambda_r} \\ &\leq 2^{\Psi_{i+1}}\lambda^{4\Psi_{i+1}}T_i^{\lambda-\Lambda_{i+1}}\Delta^{\Psi_{i+1}} \leq 2^{\Psi_{i+1}}\lambda^{4\Psi_1}T_i^{\lambda-\Lambda_{i+1}}\Delta^{\Psi_1},\end{aligned}$$

where we used that $\Lambda_{i+1} \geq \Psi_{i+1} \geq \lambda_r$, since k is large enough with respect to ε (and thus, to i^*).

This is also an upper bound on $|\mathcal{C}^{\{c,d\}}(e,\lambda)|$: since $C \setminus P$ must start at u and end at a leaf of H, once this leaf is determined and since H is a tree, C is also determined.

Recall that if a cycle C is not $\{c,d\}$-significant, then $\Pr\left(A_C^{\{c,d\}}\right) = 0$, and it does not contribute to the final sum. By Lemma 6, $L_i \geq \varepsilon_1 \Delta$ and by Lemma 8, $L_i/T_i \geq 1 + \varepsilon/10$. Using the upper bound obtained in Lemma 12 we have,

$$\begin{aligned}
\sum_{C\in\mathcal{C}^{\{c,d\}}(e)} \left(1+\frac{\varepsilon}{10}\right)^{\lambda_i(C)/2} \Pr\left(A_C^{\{c,d\}}\right) &= \sum_{\lambda\geq\Lambda_i} \sum_{C\in\mathcal{C}^{\{c,d\}}(e,\lambda)} \left(1+\frac{\varepsilon}{10}\right)^{\lambda/2} \Pr\left(A_C^{\{c,d\}}\right) \\
&\leq \sum_{\lambda\geq\Lambda_i} \left(1+\frac{\varepsilon}{10}\right)^{\lambda/2} |\mathcal{C}^{\{c,d\}}(e,\lambda)| \Lambda_{i+1} \lambda^{2\Psi_1+\Lambda_{i+1}} L_i^{-(\lambda-\Lambda_{i+1}+4\Psi_1)} \\
&\leq \sum_{\lambda\geq\Lambda_i} 2^{\Psi_{i+1}} \left(1+\frac{\varepsilon}{10}\right)^{\lambda/2} \lambda^{6\Psi_1+\Lambda_{i+1}+1} \Delta^{\Psi_1} \frac{T_i^{\lambda-\Lambda_{i+1}}}{L_i^{4\Psi_1} L_i^{\lambda-\Lambda_{i+1}}} \\
&\leq \sum_{\lambda\geq\Lambda_i} 2^{\Psi_{i+1}} \left(1+\frac{\varepsilon}{10}\right)^{\lambda/2} \lambda^{6\Psi_1+\Lambda_{i+1}+1} \frac{\Delta^{\Psi_1}}{L_i^{4\Psi_1}} \left(\frac{T_i}{L_i}\right)^{\lambda-\Lambda_{i+1}} \\
&\leq \Delta^{-3\Psi_1} \sum_{\lambda\geq\Lambda_i} 2^{\Psi_{i+1}} \lambda^{6\Psi_1+\Lambda_{i+1}+1} \varepsilon_1^{-4\Psi_1} \left(1+\frac{\varepsilon}{10}\right)^{-(\lambda/2-\Lambda_{i+1})} \\
&\leq \Delta^{-3\Psi_1} \sum_{\lambda\geq\Lambda_i} 2^{\Psi_{i+1}} \lambda^{6\Psi_1+\Lambda_1+1} \varepsilon_1^{-4\Psi_1} \left(1+\frac{\varepsilon}{10}\right)^{-(\lambda/2-\Lambda_1)},
\end{aligned}$$

where we have used that $L_i \geq \varepsilon_1 \Delta$ (Lemma 8).

Let λ_0 be the smallest integer λ such that

$$2^{\Psi_{i+1}} \lambda^{6\Psi_1+\Lambda_1+1} \varepsilon_1^{-4\Psi_1} \left(1+\frac{\varepsilon}{10}\right)^{-(\lambda/4-\Lambda_1)} \leq 1.$$

This value clearly exists since the LHS of the previous inequality goes to 0 when $\lambda \to +\infty$. Observe that in such a case,

$$\sum_{\lambda\geq\lambda_0} 2^{\Psi_{i+1}} \lambda^{6\Psi_1+\Lambda_1+1} \varepsilon_1^{-4\Psi_1} \left(1+\frac{\varepsilon}{10}\right)^{-(\lambda/2-\Lambda_1)} \leq \sum_{\lambda\geq\lambda_0} \left(1+\frac{\varepsilon}{10}\right)^{-\lambda/4} = O(1).$$

Notice that the ε_1 given in Lemma 8 only depends on ε. Let Δ be large enough such that

$$\Delta \geq \Delta_0(\varepsilon,k) = \sum_{\lambda=0}^{\lambda_0-1} 2^{\Psi_{i+1}} \lambda^{6\Psi_1+\Lambda_1+1} \varepsilon_1^{-4\Psi_1} \left(1+\frac{\varepsilon}{10}\right)^{-(\lambda/2-\Lambda_1)}.$$

Then,

$$\sum_{C\in\mathcal{C}^{\{c,d\}}(e)} \Pr\left(A_C^{\{c,d\}}\right) \leq \Delta^{-3\Psi_1}(\Delta_0(\varepsilon,k) + O(1)) = O(\Delta^{-7}),$$

since $\Psi_1 \geq 3$. ∎

Lemma 16. *With positive probability, $P(i)$ holds for every $0 \leq i \leq i^*$.*

Proof. This can be proved by induction. The basis of the induction is that $P(0)$ holds, which does deterministically by Lemma 5. Now consider $i \geq 1$. By induction hypothesis, $\Pr(P(i-1)) > 0$. Thus it suffices to show that $\Pr(P(i)) > 0$, provided that $P(i-1)$ holds. We will apply Lemma 14 to $\mathcal{E}$ in order to prove that

$$\Pr(P(i)) = \Pr\left(\cap_{E\in\mathcal{E}}\overline{E}\right) > 0.$$

Let $\beta := (1+\varepsilon/10)^{1/2}$. Then $\nu = \frac{2\log 2}{\log\beta} = \frac{4\log 2}{\log(1+\varepsilon/10)}$. For a type A event E, we define its weight $w := -\log_\Delta(\Pr(E)) > 0$ and we set $h := \Delta^{-w} = \Pr(E)$. Observe also that,

$$\beta^w h = \beta^{-\log_\Delta \Pr(E)} \cdot \Pr(E) = (\Pr(E))^{1-\log_\Delta \beta} \leq \sqrt{\Pr(E)} \leq 1/2,$$

when Δ and g are large enough with respect to β.

For an event E of other type, define its weight $w := 1$ and set $h := \mathrm{e}^{-\Omega(\Delta^{1/3})}$. Then $\Pr(E) \leq h$ by Lemma 11. Also note that $\beta^w h = \beta \mathrm{e}^{-\Omega\left(\Delta^{1/3}\right)} \leq 1/2$, when Δ is large enough.

Thus, it remains to show that for every event E_r the sum $\sum_{E_s\in\mathcal{D}_r}\beta^{w_s}h_s$ is not too large. Consider the case where $E_r = A_C^{\{c,d\}}$ for some $\{c,d\}$-significant cycle C. Let us first construct the set $\mathcal{D}_r$. For any other cycle C' and pair of colours (c',d'), $A_{C'}^{(c',d')} \in \mathcal{D}_r$ if and only if C and C' intersect in at least one free edge. For an event E_s of type B_e, $C_{v,c}$ or $D_{v,c}$, $E_s \notin \mathcal{D}_r$ if it corresponds to vertices or edges at distance at least 4 from any free edge of C.

Recall that, by the second part of Lemma 12, we have that if $E_r = A_C^{\{c,d\}}$ for some cycle C, then $\omega_r \geq \lambda_i(C)/3$. We use Lemma 15 on each free edge $e \in E(C)$ in order to bound the following sum

$$\begin{aligned}
\sum_{E_s\in\mathcal{D}_r}\beta^{w_s}h_s &= \sum_{(c',d')}\sum_{\substack{e\in E(C)\\ e \text{ free}}}\sum_{C'\in\mathcal{C}^{(c',d')}(e)}\beta^{w_s}\Pr\left(A_{C'}^{(c',d')}\right) + \sum_{\substack{E_s\in\mathcal{D}_r\\ E_s \text{ not type } A}}\beta^{w_s}h_s \\
&\leq \sum_{(c',d')}\sum_{\substack{e\in E(C)\\ e \text{ free}}} O(\Delta^{-7}) + \sum_{\substack{E_s\in\mathcal{D}_r\\ E_s \text{ not type } A}}\beta\mathrm{e}^{-\Omega\left(\Delta^{1/3}\right)} \\
&= \lambda_i(C)O(\Delta^{-5}) + \lambda_i(C)O\left(\Delta^4\mathrm{e}^{-\Omega(\Delta^{1/3})}\right) \\
&= \lambda_i(C)O(\Delta^{-5}) \leq \frac{w_r}{\nu}.
\end{aligned}$$

Now consider an event $E_r = B_e$. For some event $A_{C'}^{(c',d')}$ to depend on E_r, the cycle C' must go through a free edge within distance 4 to e. Therefore

$$\begin{aligned}
\sum_{E_s\in\mathcal{D}_r}\beta^{w_s}h_s &= \sum_{\substack{f \text{ within}\\ \text{distance 4 of } e}}\sum_{(c',d')}\sum_{C'\in\mathcal{C}^{(c',d')}(f)}\beta^{w_s}\Pr\left(A_{C'}^{(c',d')}\right) + \sum_{\substack{E_s\in\mathcal{D}_r\\ E_s \text{ not type } A}}\beta^{w_s}h_s \\
&\leq \sum_{\substack{f \text{ within}\\ \text{distance 4 of } e}}\sum_{(c',d')} O(\Delta^{-7}) + \sum_{\substack{E_s\in\mathcal{D}_r\\ E_s \text{ not type } A}}\beta\mathrm{e}^{-\Omega\left(\Delta^{1/3}\right)} \\
&= O(\Delta^{-1}) + O(\Delta^4\mathrm{e}^{-\Omega(\Delta)}) \leq \frac{w_r}{\nu}.
\end{aligned}$$

The corresponding inequalities for events of type $C_{v,c}$ or $D_{v,c}$ can be shown in a similar way. Hence, it follows from Lemma 14 that $P(i)$ is satisfied with positive probability. ∎

Proof of Lemma B. From Lemma 16, $P(i^*)$ holds with positive probability. Thus, the lemma follows directly from Lemma 4. ∎

6. PROOF OF LEMMA C

In this section we prove that the partial colouring obtained at the end of the iterative colouring procedure can be extended to a complete acyclic edge colouring of G.

Proof of Lemma C. The idea of the proof is to assign to each uncoloured edge a colour chosen uniformly at random from its list and to show that with positive probability, the obtained colouring satisfies the desired properties.

Let $L := \gamma\Delta$ and $T := \gamma^2\Delta/128$. For every two adjacent uncoloured edges e,f and every colour c, let $B^c_{e,f}$ denote the event that e,f are both assigned colour c. Consider a total order of the set of edges $E(G)$. For every cycle C and every pair of colours $\{c,d\}$, let $A_C^{\{c,d\}}$ denote the event that C is bicoloured using the pair of colours $\{c,d\}$, such that the smallest edge in C is coloured with c. Let $\mathcal{E}$ denote the set of these events.

Let $p =: 1/L$. If an event E_r is of type B, let $w_r := 2$. Then,

$$\Pr(E_r) \le \left(\frac{1}{L}\right)^2 = p^{w_r}.$$

For every cycle C, we denoted by $\lambda(C)$ its number of uncoloured edges. If an event E_r is of type A and corresponds to a cycle a C, let $w_r := \lambda(C)$. As before,

$$\Pr(E_r) \le \left(\frac{1}{L}\right)^{\lambda(C)} = p^{w_r}.$$

By the hypothesis of the lemma, every cycle that can become completely bicoloured using the pair of colours $\{c,d\}$, has at least 3 uncoloured edges. Then, for every uncoloured edge e, every pair of colours $\{c,d\}$ and every $\lambda \ge 3$, there are at most $T^{\lambda-2}$ cycles C that contain edge e and that can become bicoloured. For the cycles that cannot become bicoloured, we have $\Pr\left(A_C^{\{c,d\}}\right) = 0$, and they do not have any contribution. As in Lemma 15, we observe that when we construct a cycle starting at the edge e, at every step we either select a coloured edge with the right colour (at most one option) or we select one of the at most T different uncoloured edges that have the right colour on their reserved set of colours. The -2 in the exponent is standard in these type of countings, and comes from the fact that e is uncoloured and that the last uncoloured edge selected in the cycle has only one way to be chosen instead of T.

As there are in total at most $(1+\varepsilon)\Delta$ colours and an uncoloured edge can choose only L of them, there are at most $(1+\varepsilon)\Delta L < 2\Delta L$ choices of colours $\{c,d\}$ such that $\Pr\left(A_C^{\{c,d\}}\right) > 0$ for some cycle C going through e.

Notice that if two cycles do not share an uncoloured edge, their corresponding type A events are independent. Also note that the colour an edge is assigned, depends on at most $2LT$ events of type B. Let $\mathcal{C}^{\{c,d\}}(e,\lambda)$ be the set of cycles C that contain edge e that can be bicoloured using the pair of colours $\{c,d\}$. Therefore, if $E_r = A_C^{\{c,d\}}$ and $\mathcal{D}_r$ is a set of events such that $\mathcal{E} \setminus \mathcal{D}_r$ is mutually independent from E_r, then

$$\begin{aligned}
\sum_{E_s\in\mathcal{D}_r}(2p)^{w_s} &\le \sum_{\substack{e\in E(C):\\ e\text{ uncoloured}}}\left(\frac{2}{L}\right)^2 2LT + \sum_{\substack{e\in E(C):\\ e\text{ uncoloured}}}\sum_{(c',d')}\sum_{\lambda\ge 3}\sum_{\substack{C'\in\mathcal{C}^{(c',d')}(e,\lambda):\\ \Pr\left(A_{C'}^{(c',d')}\ge 0\right)}}\left(\frac{2}{L}\right)^{\lambda}\\
&\le \lambda(C)\left(\frac{2}{L}\right)^2 2LT + \lambda(C)\cdot 2\Delta L\sum_{\lambda\ge 3}T^{\lambda-2}\left(\frac{2}{L}\right)^{\lambda}\\
&= 8\lambda(C)\cdot\frac{T}{L} + 8\lambda(C)\cdot\frac{\Delta}{L}\sum_{\lambda\ge 3}\left(\frac{2T}{L}\right)^{\lambda-2}\\
&< \frac{\gamma\lambda(C)}{32} + \frac{8\lambda(C)}{64-\gamma} < \frac{\lambda(C)}{2} = \frac{w_r}{2}.
\end{aligned}$$

If $E_r = B^c_{e,f}$ for a pair of edges (e,f) and a colour c, then it is independent to all but $4LT$ events of type B. Therefore

$$\begin{aligned}
\sum_{E_s\in\mathcal{D}_r}(2p)^{w_s} &\le \left(\frac{2}{L}\right)^2 4LT + \sum_{(c',d')}\sum_{\lambda\ge 3}\sum_{\substack{C'\in\mathcal{C}^{(c',d')}(e,\lambda)\cup\mathcal{C}^{(c',d')}(f,\lambda):\\ \Pr\left(A_{C'}^{(c',d')}\ge 0\right)}}\left(\frac{2}{L}\right)^{\lambda}\\
&\le \left(\frac{2}{L}\right)^2 4LT + 2\Delta L\sum_{\lambda\ge 3}2T^{\lambda-2}\left(\frac{2}{L}\right)^{\lambda}\\
&= 16\cdot\frac{T}{L} + 16\cdot\frac{\Delta}{L}\sum_{\lambda\ge 3}\left(\frac{2T}{L}\right)^{\lambda-2}\\
&\le \frac{\gamma}{8} + \frac{16}{64-\gamma} < 1 = \frac{w_r}{2}.
\end{aligned}$$

We use the original version of the Weighted Lovász Local Lemma [16, p. 221] to conclude that, with positive probability, none of the events in $\mathcal{E}$ happen. ∎

Now, let $\{S_v : v\in V(G)\}$ be the collection of subsets of $[(1+\varepsilon)\Delta]$ that satisfies properties (A.1)–(A.3) whose existence is proved in Lemma A. Let χ be a partial edge colouring of G with properties (B.1)–(B.3) that is guaranteed to exist by Lemma B using this collection of sets. Let $\gamma := \frac{\varepsilon^2}{18}$. Assign to each uncoloured edge $e = uv$ in χ a list of $\gamma\Delta$ colours chosen arbitrarily from $S_u\cap S_v$. This is possible since by property $(A.2)$ in Lemma A, these sets have size at least $\gamma\Delta$. Then, properties (B.1)–(B.3) of χ imply that all the hypotheses of Lemma C are satisfied and we can extend χ into a complete acyclic edge colouring of G, which finishes the proof of Theorem 1.

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